How Should We Best Account for the Adjectival Form of Number Words?

In familiar natural languages we are able to say “there are 12 cars”, but also “12 is divisible by 3”. The first (the ‘adjectival’ use) sounds as if the group of cars has an attribute (perhaps a ‘property’) of twelveness; the second (the ‘substantival’ use) sounds as if we are discussing the characteristics of an object named ‘12’. Should one these two usages have priority?

Frege approached the problem in precisely this linguistic way. Although he had an almost a priori commitment to numbers as objects, he respected the view that numbers are attributives precisely because of ordinary usage. In his *Grundlagen*, particularly §57-60, he offers brief reasons why the attributive view should be rejected. The main force of his objections comes with his later view that numbers must actually be objects.

Is the logical form of number sentences best represented as predicates attached to variables, or as quantifiers pointing to the domain of the variables, or as the singular terms giving values to the variables? If we say ‘Gordon is happy’, the form might appear as a predicate (‘Happy’) assigned to an object (‘Gordon’), so we might write *Hg*. However, ‘Gordon is alone’ might be represented as ‘Gordon is the only one present’ - perhaps as $Pg \ \& \ \forall y(Py \rightarrow y = g)$. In other words, while most adjectives seem best represented as predicates, some seem better represented using quantification and identity. English also has an easy capacity for turning adjectives into nouns, so that we can rephrase as ‘Gordon is imbued with happiness’, or ‘solitude is Gordon’s lot’, inviting us to quantify over abstract objects.

Frege’s critique is initially addressed to the possibility that adjectival number words might indicate properties. His contextual principle (1884: p.x) says that meaning only truly exists at the level of the sentence. So the meaning of an attributive word is the contribution it makes to a larger meaning, the truth-conditions of the full sentence. He assumes that an adjective such as ‘green’ will function as a predicate, as in *Gx*, which only fulfils its semantic destiny when an interpretation gives a value to x, such as ‘field’. He then observes (§58) that while ‘green’ adds something to the concept of ‘field’, the ‘one’ in ‘one green field’ does no such thing, so ‘one’ does not look like an attribute. Earlier (§30) he had made the point that numbers don’t seem to be properties if some objects can be either ‘one’ or ‘not-one’, depending on how they are viewed; a pair of boots has the ‘property’ of both one *and* two. He also claims that it is characteristic of attributes to make us imagine features of objects, which numbers fail to do, especially in the case of the number ‘zero’. While a failure to make us imagine features of objects might cramp our ability to think about entities like fields, this is irrelevant to logical reasoning or arithmetic. Finally he claims (§57) that while number terms are attached to concepts rather than to objects, they are still not properties of those concepts, because they only function as *part* of the predicate.

His solution is to claim that not only can ‘Jupiter has four moons’ (adjectival use) be rewritten as ‘the number of Jupiter’s moons is four’ (substantival use), but that the latter reading is *correct*. Thus the logical form of the numerical part of the sentence is not a quantifier or a predicate, but a singular term, denoting an object. However, translations can easily move in the opposite direction (presenting singular terms adjectivally), and Dummett observes that Frege gives us no reason for preferring his direction of translation. After all, ‘the number of Jupiter’s moons is decreasing’ doesn’t mean that *four* is decreasing (Yablo 2000:219). Frege’s claim looks stronger when arithmetic moves into pure abstract reference, where we can say ‘there is one prime number between 13 and 19’ (which resembles ‘pink is more like red than it is like green’), and defenders of the adjectival use must account for this usage, which seems basic to arithmetic. In Frege’s view the adjectival use is closer to the *application* of arithmetic; to say there are ‘four moons’ connects the number-concept to the physical world, but ‘the number of moons is four’ gives us the underlying mathematical fact. One might compare ‘he is a London boy’, where two objects, a boy and London, are connected, using an adjectival syntax. Frege also makes the interesting claim (§56) that while attributive accounts might give us the sense of a number word, they will not provide a reference, so we lack identity conditions between two numbers, and might even identify a number with the wrong type of object.

Dummett says that Frege's suggestions for attributive definitions of 0, 1, and n would nowadays be presented as 'numerically definite quantifiers', so that 'Jupiter has four moons' might be written as $\exists_4x(Mx)$, where the suffix '4' is shorthand for a string of four existential quantifiers. This strategy for representing arithmetic is quickly dismissed by Frege, but numerical quantifiers seem comfortably at home with other quantity adjectives such as 'no', 'few', 'numerous', 'many', 'innumerable'. But while the logical form of adjectival numbers may be such strings of quantifiers, this makes the logical form of '780,976 Fs' very bizarre, and the logical form of 'an infinity of Fs' becomes inconceivable. However, it at least seems prima facie plausible that number words might be a large family of specific quantifiers, given that they blatantly specify quantities. When Frege tells us that a key step is to see 'there are as many Fs as there are Gs' as a one-to-one correlation, the word 'many' sounds thoroughly adjectival, and Rumfitt observes (2002:49) that 'Jupiter has many moons' does not invite the translation 'the number of Jupiter's moons is identical with the number many'.

Hofweber (2005) also begins with natural language, and the question of priority between adjectival and substantival uses of number terms. He identifies three responses to the adjectival use: that number words are **ambiguous** between the two uses (rejected because both usages seem to refer to the same set of truths, and because *systematic* ambiguity, one for each number, seems implausible); that the adjectival use is an **unimportant** quirk of language irrelevant to scientific usage (rejected as the adjectival use might have scientific utility, and there appears to be a real difference between the usages); and the possibility that the adjectival use is **syncategorematic** – that the natural language number attributes can 'disappear in analysis', as Russell eliminated 'the' from the logical form of definite descriptions.

This last view must be taken more seriously. The standard analysis of adjectival words, if they are syncategorematic (outside the normal categories of meaningful units, but relating such things together), is as numerically definite quantifiers. But this produces huge blocks of quantifiers, and also means that each natural number will have a different logical form. Does 'there are six oranges' have an different logical form from 'there are seven oranges'? Hofweber also challenges the analogy with Russell's treatment of 'the'; in that case 'the' disappears into a set of quantifiers, predicates and identities, but the syncategorematic approach of Frege seeks to eliminate the adjectival usage while *retaining* the singular-term substantival usage. Thus the proposed analysis gives an account of the semantics (as blocks of quantifiers), but won't explain the two different forms of syntax. Hofweber points out that in the analytical approach (to natural language terms indicating quantity), 'the' can be eliminated, while 'some' survives among the quantifiers, but 'many' is inexpressible in first-order predicate calculus. A unified approach to such terms seems to be desirable, and Hofweber sees it in the semantics and 'generalised quantifier theory' of Mostowski and Montague.

Atomic noun phrases are broken apart, explaining the 'semantic value' of the components, which are a noun and a 'determiner' (typically an adjective). The whole sentence gets its semantic value when the function of the compiled noun phrase is applied to a verb phrase. This compositional approach means that the determiners (including adjectives) will not be analysed away, but will contribute to the semantics of sentences. The determiners 'the' and 'some' and 'many', and a wide range of other natural language quantifiers, can fall comfortably within this account, including complex determiners linked by Boolean operators.

However, we now have an account of the semantic value of adjectival number words, but one which dispenses with the singular terms, while Frege and others dispense with the adjectives but retain the singular terms, so Hofweber pursues a unified account of both usages.

First he notes two usages in standard English: the anaphoric use of number adjectives (as in 'we ate three cakes, so two were left', where 'two' is clearly adjectival, despite the dropping of its noun), and the abstract reference usage (as in 'some is more than none', or 'three is more than two'). In each case the result is a 'bare determiner', which seems to be a free-standing adjective. The first case he calls 'elliptically bare', and it has a contextual input; the second he calls a 'generalisation', which is 'semantically bare'.

In the construction 'not many, but enough', the Boolean word 'but' (or 'and') does not concatenate sentences, but builds collective concepts from complex quantifiers. This corresponds to the ‘collective’ and ‘distributive’ uses of words for groups, as when we say ‘the team *is* happy’ when they score a goal, but ‘the team *are* happy’ when they take a summer break. If we combine bare determiners with Boolean operators to say ‘two and two are four’, then arithmetic floats to the surface of ordinary English. The phenomenon that people are divided over whether the symbolic statement ‘2+2=4’ should be articulated as ‘two and two *are* four’ or ‘two and two *is* four’ suggests that while the former is the language of bare determiners, the latter asserts an identity between two objects, illustrating a slide from adjectival to substantival, and suggesting an approach to Frege’s point about identity. Hofweber notes a common human tendency to shift to substantival syntax (which psychologists call ‘cognitive type-coercion’) when faced with tasks in reasoning.

Speakers also like to emphasise key aspects of each spoken sentence. In addition to emphasis by volume, speakers use the ‘clefted sentence’, which offers an unorthodox word-order to shift the verbal focus, as when we say ‘orange is the colour of his shirt’ rather than ‘he is wearing an orange shirt’ (when discussing colours). In that light, ‘the number of Jupiter’s moons is four’ sounds more like a rearrangement for a context than the logical form of the thought.

Having shown how the ordinary adjectival usage of number words can lead to simple arithmetic, the problem of infinities is dealt with. The formulation of adjectival number statements as second-order logic, using numerically definite quantifiers, requires an infinite supply of objects, to avoid the consequence that statements involving cardinalities greater than the number of objects become vacuously true. Axiomatisations of arithmetic frequently assert the existence of such a supply. Hofweber notes (but rejects, as obscure) the possibility of a modal account, but prefers to avoid the problem by denying the existence of objects. He offers the example that while ‘two dogs are more than one’ seems to require the existence of dogs, ‘two unicorns are more than one’ seems true without unicorns. He is sympathetic to the extreme generality found in a logicist view, but not with an ontology of objects, because he distinguishes the use of quantifiers to refer to a domain of reference from the use to establish inferential relations. The former seems to require an ontology of objects, but the latter does not. The two uses often coincide, but can come apart; ‘there is something in the fridge, so we’re fine’ seems committed to food in the fridge, but ‘if there is something in the fridge then we are fine’ does not.

Hofweber’s object-free view of logicism avoids one of its major problems. Hodges observes (2001:9) that modern logic does not chart the world of reason, but is just a formal activity exploring certain artificial languages, so no ontology can be inferred. Indeed, it seems unlikely that the disappearance all concrete or abstract objects from reality would change the nature of logical consequence.

Hofweber succeeds in giving an account of arithmetical language which gives primacy to the adjectival usage. Central to his theory is his account of quantifiers, semantically constructed from adjectival usage, and then playing a purely inferential role, rather than asserting reference. As long as ‘quantifying over’ is a euphemism for commitment to existence, the case for numerical objects becomes strong, but if quantifiers can be read differently, that case is weakened. However, scepticism about numerical objects is a very reasonable starting point; they seem to lack causal power, and Yablo criticises metaphysicians for a tendency to discovery ‘unexpected objects’ in almost every area of their subject, citing countermodels, worlds, functions, numbers, events, sets and properties as his examples (2000:198).

Hofweber’s logicism has the attractions of extreme generality (Frege’s reason for its wide applicability) and indifference to ontology. It has, however, a difficulty which Frege avoided. The subject-matter of arithmetic can be clearly delineated if it is about certain objects, which have individuation conditions, can be derived from simple axioms, and can be identified with one another. But without objects it is not clear how arithmetic can be demarcated from the rest of logic. It won’t be enough to say that arithmetic is the aspect of logic that involves quantification, since arithmetic involves precision, rather than quantitative generalisations. A logicist account will struggle to explain the phenomenon of counting, and will not explain the way in which the numerical size of an entity can behave like a property (as the size of a symphony orchestra is the best explanation of its aural power).

The strongest reason for thinking that number words are essentially adjectival is the fact that they fit comfortably into a family of general quantity words. Thus the sequence *no/zero ..a/one ..several ..eight ..many ..fifteen ..numerous ..132 ..innumerable....* seems to an English-speaker to be a natural group, containing no category mistakes. The word ‘large’ seems to be self-evidently adjectival when attached to a flock of sheep, and making the large number precise seems to communicate the same sort of information (as in ‘a 228-strong flock’).

So we must distinguish the precise quantity-adjectives from the vague ones, but allowing a continuum from vague (‘large’), to fairly precise (‘just over 200’), to exact (‘228’). Frege must be right that concepts are involved, since whether a flock is large is too relative to be a direct observation, and we must know if we are counting sheep or their feet. The crucial observation, though, (and this could apply equally to abstracta as to concreta) is that a flock has properties because it is a whole composed of parts. The realisation of the importance of numerical adjectives leads us to the sort of mereological explorations initiated by David Lewis (e.g. 1993).

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